Time Series Econometrics
Assignment 4

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1. Let L be the lag operator.

   (a) Let $\phi(L) = 1 - 0.75L$ and $\phi^{-1}(L) = \sum_{i=0}^{\infty} a_i L^i$. Calculate a_i for $i = 0, 1, 2 \ldots$.

   (b) Let $\phi(L) = 1 - 0.5L + 0.06L^2$ and $\phi^{-1}(L) = \sum_{i=0}^{\infty} a_i L^i$. Calculate a_0, a_1, a_2, a_3, a_4.

2. Suppose $Y_t = \phi(L)\varepsilon_t$ where $\phi(L)$ is defined as in Problem 1(a) and $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$. Obtain the asymptotic distribution of $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} Y_t$ as $T \to \infty$.

3. Let $\{X_t\}$ be a weakly stationary time series, and denote its autocovariance function by $\gamma_X(k)$. Suppose that $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$. The spectral density of $\{X_t\}$, denoted by $f_X(\lambda)$, is a real function on $[-\pi, \pi)$ such that

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\lambda k},$$

   where i is the imaginary unit defined by $i^2 = -1$. (If you have difficulties understanding this and the corresponding exponential function, you should search online and get yourself familiar with the algebra of the exponential functions.)

   (a) What is $f_X(0)$?

   (b) Show that

$$\gamma_X(k) = \int_{-\pi}^{\pi} e^{i\lambda k} f_X(\lambda) \mathrm{d}\lambda.$$

   (c) Let $\{X_t\}$ and $\{Z_t\}$ be two weakly stationary time series with spectral densities $f_X(\lambda)$ and $f_Z(\lambda)$, respectively. Suppose that

$$Z_t = \sum_{j=-\infty}^{\infty} \phi_j X_{t-j}.$$

   Show that

$$f_Z(\lambda) = \left| \sum_{j=-\infty}^{\infty} \phi_j e^{-i\lambda j} \right|^2 f_X(\lambda).$$

   where for a complex number $x = a + bi$, $|x| = (a + bi)(a - bi)$.

(d) Let $Y_t = \sum_{i=0}^{\infty} \phi_i \varepsilon_{t-i}$ where $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$ and $\sum_{i=0}^{\infty} |\phi_i| < \infty$. Obtain the spectral density of $\{Y_t\}$.

4. Obtain the autocorrelation function $\rho(\cdot)$ of the ARMA$(1,1)$ process

$$X_t - \phi X_{t-1} = \varepsilon_t + \theta \varepsilon_{t-1}$$

where $|\phi| < 1, |\theta| < 1$, and $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$. If the ε_t's are iid, obtain the limit distribution of $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} X_t$.